MODELLING OF FINGERING PHENOMENON IN HETEROGENEOUS POROUS MEDIA

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ABSTRACT

In this article, the phenomenon of fingering in a particular displacement method concerning two immiscible fluids through a dipping heterogeneous porous medium with mean capillary pressure has been discussed under certain conditions. This phenomenon gives a nonlinear partial differential equation as a governing equation, which is solved by ho-motopy perturbation sumudu transform method (HPSTM). The oil recovery rate for the fingering phenomenon in heterogeneous porous media is calculated and its graph is increasing, which is feasible with the fact of the oil recovery process. Thus HPSTM is a very effective method as compared to the existing standard methods to handle nonlinear partial differential equations.

Key Words: Fluid flow through porous media; Fingering phenomenon; Similarity transformation; homotopy perturbation sumudu transform method (HPSTM).

1 INTRODUCTION:

The fingering (instability) phenomenon [9] is an important phenomenon of petroleum technology [12], where water drives are employed for the recovery of oil. In fact, the fingers are the discontinuities arising on the smooth common displacement front. Buckley and Levrett [1] discussed this problem without considering capillary pressure. While the other authors like Fayers [4], Patel et al. [8], Scheidegger-Johnson [9] and Verma [14, 15] discussed it from different viewpoints. For example, Mehta et al. [7] gave the numerical solution of this problem with capillary pressure effect and Choksi et al. [3] found an approximate solution of this problem arising in homogeneous porous media by Successive Linearisation method. Assume that the individual pressure of the two flowing phases can be replaced by their mean capillary pressure [4] and an expression for phase saturation distribution is obtained. The mathematical formulation leads to a nonlinear partial differential equation. Also, the injection of water into an oil formation in porous medium is furnishing a two-phase liquid-liquid flow problem. Generally, such problem is encountered in the secondary oil recovery process of petroleum technology [13], replenishment problem of groundwater hydrology, geophysics, reservoir engineering [2], etc. The main goal of the present article is to obtain a solution of this phenomenon.
in heterogeneous porous media using homotopy perturbation sumudu transform method (HP-STM) [11], which is an elegant combination of sumudu transformation, homotopy perturbation method and He’s polynomial. HPSTM is a very effective method to handle nonlinear partial differential equation.

2 STATEMENT OF THE PROBLEM:

Water (w) is injecting uniformly with constant velocity into a finite cylindrical piece of a heterogeneous porous medium of length L, which is completely saturated with a native fluid-oil (o). This gives a well-developed finger flow, which is called the fingering (instability) phenomenon (Figure 1). Also, \( x = 0 \) (x is measured in the direction of displacement) is called the initial boundary and because of the effect of injecting water the oil on the initial boundary is displaced through a small distance. The cylinder is totally surrounded by an impermeable surface except its initial boundary \( x = 0 \) as shown in Figure 2. Figure 2 shows the schematic demonstration of the fingering phenomenon [8].

![Figure 1: Fingering phenomenon during oil recovery process](image1.png)

![Figure 2: Schematic diagram of the fingering phenomenon](image2.png)
3 STATICs OF FINGERS:

Scheidegger-Johnson [9] considered an average cross-sectional area only occupied by the fingers. It shows a saturation of water $S_w(x, t)$ at injected water level $x$ with time $t$ in the porous medium (Figure 3).

Scheidegger-Johnson [9] gave the following relationship

$$k_w = S_w, \quad k_o = S_o = 1 - S_w$$  \hspace{1cm} (1)

Figure 3: Average cross-sectional area for the fingering phenomenon

4 FUNDAMENTAL EQUATIONS:

By Darcy’s law [13], the filtration velocity of water ($V_w$) and oil ($V_o$) can be given as

$$V_w = -\frac{k_w}{\mu_w} \frac{\partial P_w}{\partial x}$$  \hspace{1cm} (2)

$$V_o = -\frac{k_o}{\mu_o} \frac{\partial P_o}{\partial x}$$  \hspace{1cm} (3)

where $k = k(x)$ is the variable because the medium is heterogeneous.

The equations of continuity are

$$\phi \frac{\partial S_w}{\partial t} + \frac{\partial V_w}{\partial x} = 0$$  \hspace{1cm} (4)

$$\phi \frac{\partial S_o}{\partial t} + \frac{\partial V_o}{\partial x} = 0$$  \hspace{1cm} (5)

where $\phi = \phi(x)$ is the variable because the medium is heterogeneous.

Consider the phase densities as constant and the porous medium is completely saturated, thus

$$S_w + S_o = 1$$  \hspace{1cm} (6)
5 CAPILLARY PRESSURE:

In two phase fluid flow, the capillary pressure \( P_c \) is the pressure difference across the interface between oil and water. Also, it is a function of phase saturation [2]. So consider a continuous linear function defined as

\[ P_c = -\beta S_w \quad \text{and} \quad P_c = P_o - P_w \quad (7) \]

6 THE EQUATION OF MOTION FOR SATURATION:

By the equations (2-5), we get

\[ \phi \frac{\partial S_w}{\partial t} = \frac{\partial}{\partial x} \left( k_w \frac{\partial P_w}{\partial x} \right) \quad (8) \]

\[ \phi \frac{\partial S_o}{\partial t} = \frac{\partial}{\partial x} \left( k_o \frac{\partial P_o}{\partial x} \right) \quad (9) \]

Removing \( P_w \) from the equations (7) and (8), we get

\[ \phi \frac{\partial S_w}{\partial t} = \frac{\partial}{\partial x} \left( k_w \frac{\partial P_o}{\partial x} - \frac{\partial P_c}{\partial x} \right) \quad (10) \]

By the equations (6), (8) and (9), we get

\[ \frac{\partial}{\partial x} \left( k_w \frac{\partial P_o}{\partial x} - \frac{\partial P_c}{\partial x} \right) = 0 \quad (11) \]

\[ \therefore \frac{\partial P_o}{\partial x} = \frac{C}{k_w} + \frac{1}{\mu_w} \frac{\partial P_c}{\partial x} \quad (13) \]

By the equations (10) and (13), we get

\[ \phi \frac{\partial S_w}{\partial t} + \frac{\partial}{\partial x} \left( \frac{k_w \mu_w}{\mu_o k_w} \frac{\partial P_c}{\partial x} \right) - \frac{C}{\mu_o k_w} = 0 \quad (14) \]

The value of the pressure [14, 15] of the native fluid \( (P_o) \) is given as

\[ P_o = \bar{P} + \frac{P_c}{2}, \quad \text{where} \quad \bar{P} = \frac{P_o + P_w}{2} \quad (15) \]
\[ \therefore \frac{\partial}{\partial x} = 2 \frac{\partial}{\partial x} \]

Thus by the equation (13), we get

\[ C = \frac{1}{2} \frac{\partial P_c}{\partial x} k \frac{k_o}{\mu_o} - \frac{k_w}{\mu} \]

(17)
By the equations (14) and (17), we get
\[
\frac{\partial S_w}{\partial w} + \frac{1}{k} \frac{\partial}{\partial w} \frac{\partial P_c}{\partial S_w} = 0
\]
\[
\frac{\partial}{\partial t} + 2 \frac{\partial}{\partial x} \mu_w \frac{\partial S_w}{\partial x}
\]
(18)

Now the fictitious relative permeability is a function of water saturation. So for definiteness, consider [14]
\[
k_w = S_w, P_c = -\beta S_w
\]
(19)
\[
\frac{\partial S_w}{\partial w} = \frac{\beta}{S_w} \frac{\partial}{\partial S_w}
\]
\[
\therefore \frac{\partial}{\partial t} = 2\mu_w \frac{\partial}{\partial x} k_S \frac{\partial S_w}{\partial x}
\]
(20)

To simplify the equation (20), Chen [15] gave
\[
k \propto \phi
\]
(21)
i.e. \( k = k_c \phi \)
(22)

Substituting the above value in the equation (20), we get
\[
\frac{\partial S_w}{\partial T} = \frac{2}{\phi} \frac{\partial}{\partial X} \phi S_w \frac{\partial S_w}{\partial X}
\]
(23)

where \( S_w(0, t) = S_{wo} \) and
\[S_w(x, 0) = S_{wc} = S_{wo}e^{x/L}[8]; 0 < S_{wo} < S_{wc}\]

Using dimensionless variables
\[X = \frac{x}{L}, T = \frac{bk_c}{2\mu_w L^2} t, \text{ we get}\]
\[
\frac{\partial S_w}{\partial T} = \frac{1}{\phi} \frac{\partial}{\partial X} \phi S_w \frac{\partial S_w}{\partial X}
\]
(24)

Now by Verma [14]
\[
\phi = \frac{1}{a - bx}; (a - bx) \geq 1
\]
(25)
\[
= \frac{1}{a - bXL} \frac{b}{a - bx}
\]
(26)
\[
\therefore \frac{\partial}{\partial X} = \frac{\partial}{\partial X} (log \phi) = \frac{1}{a} = A(say)
\]
(27)

By the equations (24) and (27),
\[
\frac{\partial S_w}{\partial T} = \frac{\partial}{\partial T}
\]

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\[
\frac{\partial}{\partial X} S_w \frac{\partial S_w}{\partial X} + w \frac{\partial S_w}{\partial X} \frac{A}{S} \quad (28)
\]

where \( S_w(0, T) = S_{wo} = B \) (say) and

\[
S_w(X, 0) = S_{wc} = S_{wo} e^x = Be^x
\]
7 ANALYSIS OF HOMOTOPY PERTURBATION SUMUDU TRANSFORM METHOD (HPSTM):

To illustrate the basic idea of homotopy perturbation sumudu transform method (HPSTM) [11], consider a general nonlinear non-homogenous second order partial differential equation with the initial conditions of the form:

\[
\begin{align*}
DU (x, t) + RU (x, t) + NU (x, t) &= g (x, t); \\
U (x, 0) &= h (x), U_t (x, 0) &= f (x)
\end{align*}
\]  

Taking the sumudu transform defined by

\[
S \left[ f (x, t) ; u \right] = \int_0^\infty e^{-u} f (x, t) dt; \quad u \in (-\tau_1, \tau_2)
\]  

for any function \( f(x, t) \) on both sides of the equation (29), we get

\[
S [DU (x, t)] + S [RU (x, t)] + S [NU (x, t)] = S [g (x, t)]
\]  

Using the differentiation property of the sumudu transform and initial conditions (30) in the equation (32), we have

\[
S [U(x, t)] = u^2S [g(x, t)] + h(x) + uf(x) - u^2S [RU(x, t) + NU(x, t)]
\]  

Now, applying the inverse sumudu transform on both sides of the equation (33), we get

\[
U(x, t) = G(x, t) - S^{-1} u^2 S [RU(x, t) + NU(x, t)]
\]  

where \( G(x, t) \) represents the term arising from the source term and the prescribed initial conditions after applying inverse sumudu transform.

Now, we apply the homotopy perturbation method (HPM) [5] to solve the equation (34) by taking series form as

\[
U(x, t) = \sum_{n=0}^{\infty} p^n U_n(x, t)
\]  

where \( p \in [0, 1] \) is the perturb parameter.

The nonlinear term can be decomposed as

\[
NU(x, t) = \sum_{n=0}^{\infty} p^n H_n(U)
\]  

for some He’s polynomials \( H_n(U) \) [5] that are given by

\[
H_n(U_0, U_1, ..., U_n) = \frac{1}{n! \partial^n} \sum_{i=0}^{\#} p^n U_i \quad ; n = 0, 1, 2, 3, ... \]
Substituting the equations (35) and (36) in the equation (34), we get

\[ X \sum_{n=0}^{\infty} p^n U_n(x, t) = G(x, t) - p S^{-1} u^2 S R \sum_{n=0}^{\infty} p^n U_n(x, t) + p^n H_n(U) \]

which is the combination of the sumudu transform and the homotopy perturbation method using He’s polynomials called homotopy perturbation sumudu transform method (HPSTM).

Comparing the coefficients of like powers of p of the equation (38), the following approximations are obtained:

\[ p^0 : U_0(x, t) = G(x, t) \]
\[ p^1 : U_1(x, t) = -S^{-1}[u^2 S[RU_0(x, t) + H_0(U)]] \]
\[ p^2 : U_2(x, t) = -S^{-1}[u^2 S[RU_1(x, t) + H_1(U)]] \]
\[ p^3 : U_3(x, t) = -S^{-1}[u^2 S[RU_2(x, t) + H_2(U)]] \]

Thus the solution \( U(x, t) \) is given by taking \( p \to 1 \) as

\[ U(x, t) = U_0(x, t) + U_1(x, t) + U_2(x, t) + U_3(x, t) + \ldots \]

### 8 SOLUTION OF THE PROBLEM BY HPSTM:

To apply this method, consider the equation (28) by replacing \( S_w \) by \( S \) as

\[ S_T + ASS_X + SS_{XX} + (S_X)^2 = 0 \]

where \( S(0, T) = B \) and

\[ S(X, 0) = B e^X \]

Apply sumudu transform on the equation (44), we get

\[ S [S(X, T)] = B e^X + uS ASS_X + SS_{XX} + (S_X)^2 \]

(45) Apply inverse sumudu transform on the equation (45), we get

\[ S(X, T) = B e^X + S^{-1} uS ASS_X + SS_{XX} + (S_X)^2 \]

(46) Apply homotopy perturbation method (HPM), we get

\[ X \sum_{n=0}^{\infty} p^n S_n = B e^X + pS^{-1} uS \sum_{n=0}^{\infty} p^n H_n(U) \]

(47) where He’s polynomial \( H_n(S) \) [5] is given by
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\[ H_n (S_0, S_1, \ldots, S_n) = \frac{1}{n!} \frac{\partial^n}{\partial p^n} \prod_{i=0}^{\infty} p S_i \quad ; \quad n = 0, 1, 2, 3, \ldots \]  \hspace{1cm} (49)

Comparing like powers of \( p \), we get

\[ p^0 : S_0 = B e^X \]
\[ p^1 : S_1 = S^{-1} [uS [H_0(S)]] \]  \hspace{1cm} (50)
\[ p^2 : S_2 = S^{-3} [uS [H_1(S)]] \]
\[ p^3 : S_3 = S^{-1} [uS [H_2(S)]] \]

and so on

where He’s polynomials are given by

\[ H_0(S) = A S_0(S_0)_X + S_2(S_0)_XX + [(S_0)_X]^2 \]  \hspace{1cm} (51)
\[ H_1(S) = A[S_0(S_1)_X + S_1(S_0)_X] + [S_0S_1XX + S_1S_0XX] + 2S_0S_1X \]  \hspace{1cm} (52)

\[ H_2(S) = A[S_0(S_2)_X + 2S_1(S_1)_X + S_2(S_0)_X] + [S_0S_2XX + 2S_1(S_1XX) + S_2S_0XX] \]
\[ + 2[S_0X S_2X + (S_1X)_X]^2, \ldots \]  \hspace{1cm} (53)

Substituting all the above values and then combining all, we get the required approximate solution for the saturation of water by taking \( p \to 1 \) as

\[ S_w(X, T) = S(X, T) = B e^X + (A + 2)B^2 e^{2X} T + \frac{3}{2}(A + 2)(A + 3)B^3 e^{3X} T^2 \]
\[ + \frac{3}{3}(A + 2)(5A^2 + 41A + 98)B^4 e^{4X} T^3 + \ldots \]  \hspace{1cm} (54)

9 OIL RECOVERY:

The fraction \( \alpha \) of oil recovered to time \( T \) [6] is defined in general as

\[ \alpha(T) = \frac{V(T)}{V_{lo}} \]  \hspace{1cm} (55)

where \( V(T) = \frac{1}{L} \int_{0}^{Z} S_w(X, T) dX \)  \hspace{1cm} (56)

Substituting the value of \( S_w \) and dimensionless time \( T \) in the above equation (56), the value of the fraction \( \alpha \) of oil will be recovered to the dimensionless time \( T \).
10 NUMERICAL AND GRAPHICAL REPRESENTATION:

The numerical and the graphical representation of solution of the equation (28) for the saturation of injected water in heterogeneous porous media have been discussed using MATLAB. Figures 4 and 5 represent the graph of saturation of water $S_w(X, T)$ versus distance $X$ and for fixed time $T = 0.0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0$ respectively. Table 1 indicates the numerical values of saturation of injected water upto 9-decimal places with $S_{wo} = 0.1$.

Figure 6 shows the graph of the fractional oil recovery $\alpha$ versus dimensionless time $T$, which is increasing. So oil recovery increases with time during the secondary oil recovery process.

Table 1: Saturation of injected water $S_w(X, T)$ in heterogeneous porous media

<table>
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<tr>
<th>$T$</th>
<th>$X=0.0$</th>
<th>$X=0.1$</th>
<th>$X=0.2$</th>
<th>$X=0.3$</th>
<th>$X=0.4$</th>
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<tr>
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<tr>
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11 CONCLUSION:

By the HPSTM, the saturation of injected water during secondary oil recovery process in heterogeneous porous media for any distance X and any time $T > 0$ can be found. Looking to the graph, the saturation of injected water is increasing exponentially for small change in distance X and for all time T, which is feasible with the physical phenomenon, i.e. more oil can be produced during oil recovery process. Also, the oil recovery rate of the reservoir is increasing with time, which gives a rise of oil amount during the recovery process. So we can say that HPSTM is much better than the other existing standard methods to solve nonlinear partial differential equation.

![Figure 4: Saturation of injected water $S_w(X, T)$ versus distance X for fixed time T by HPSTM](image)

![Figure 5: Saturation of injected water $S_w(X, T)$ versus time T for fixed distance X by HPSTM](image)
Fractions oil recovery $\alpha$ versus time $T$

REFERENCES


